## CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD



# Restricted Symmetric Collinear 

Central Configuration for Six-Body
by

## Mohsin Ali

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the<br>Faculty of Computing<br>Department of Mathematics

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First of all, I dedicate this research project to Allah Almighty, The most merciful and beneficent, creator and Sustainer of the earth and

Dedicated to Prophet Muhammad (peace be upon him) whom, the world where we live and breathe owes its existence to his blessings
and
Dedicated to my grand father Mian Muhammad Siddique Peerzada and

Dedicated to my parents and Siblings, who pray for me and always pave the way to success for me and

Dedicated to my teachers, who are a persistent source of inspiration and encouragement for me

## CERTIFICATE OF APPROVAL

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## Abstract

This thesis is consists of two parts. In first part the collinear central configuration for five masses is discussed. In the second part, the equation of motion of $6^{\text {th }}$ particle is considered in the gravitational field of five masses. After finding the equation of motion of $6^{\text {th }}$ particle the equilibrium points and their linear stability analysis is examined by using Mathematica. In the last part the permissible region of motion is explored for test particle by using different values of Jacobian constant.

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## Abbreviations

| $C C$ | Central Configuration |
| :--- | :--- |
| $\mathbf{M s}$ | Mass of the Sun |
| $\mathbf{R}$ | Region |
| $\mathbf{R 5 B P s}$ | Restricted Five-Body Problems |
| $\mathbf{S I}$ | System International |
| $\mathbf{2 B P s}$ | Two-Body Problems |
| $\mathbf{3 B P s}$ | Three-Body Problems |
| $\mathbf{4 B P}$ | Four-Body Problem |
| $\mathbf{5 B P s}$ | Five-Body Problems |

## Symbols

| Symbol | Name | Unit |
| :--- | :--- | :--- |
| $\mathbf{F}$ | Gravitational force | Newton |
| $G$ | Universal gravitational constant | $\mathrm{m}^{3} \mathrm{~kg}^{-1} \mathrm{~s}^{-2}$ |
| $\mathbf{L}$ | Angular momentum | $\mathrm{kgm}^{2} \mathrm{~s}^{-1}$ |
| $\mathbf{p}$ | Distance | Meter |
| $\boldsymbol{\rho}$ | Linear momentum | $\mathrm{kgms}^{-1}$ |
| $\ni$ | Such that |  |
| $\forall$ | For all |  |
| $\in$ | Belongs to |  |

## Chapter 1

## Introduction

The $n$-body problem in mechanics is the problem of determining the individual motions of a group of celestial objects that interact gravitationally towards each other. The purpose behind resolving these sort of problems is to know about the motion of the moon, the sun, planets, visible stars etc. In the 17 th century mathematicians and astronomers were attracted to $n$-body problem. Isaac Newton resolved two body-problem (2BP) through his laws of motions and the universal law of gravity. There is no significant way to solve the problem if $n \geq 3$, but if we have a restricted $n$-body problem it may provide a particular solution. Mathematicians and astronomers have continued working on the $n$-body problem during the last four centuries. First, in the 17th century, Kepler defined the elliptical trajectories of planets around the sun in his planetary laws of motion between 1609 and 1619 "Philosophiae Naturalis Principia Mathematica" [1].

One of the most important works in the history of science, in which Newton derived and formulated Keplers law. As a special case, the law for two particles when they are interacting with each others by gravitational force is:

$$
\begin{equation*}
\mathbf{F}=G \frac{m_{1} m_{2}}{r^{3}} \mathbf{r} \tag{1.1}
\end{equation*}
$$

where the two masses $m_{1}$ and $m_{2}$ are apart from each others by $r$ and $G$, is the universal gravitational constant. After the justification of Keplers laws, Newton
turned his attention to comparatively more complex systems. Alexis Clairaut succeeded in presenting an approximation for the 3BP. After some small adjustment, his work accounted for the perigee of the moon, which was the aim of Newton. He won the St. Petersburg Academy prize in 1752. When Halleys comet passed by earth in 1759, the value of his approximations was amply to demonstrate its motion. He himself take off the margin of error which he predicted in his equations, within a month. Leonhard Euler also work on the 3BP at the same time. The extremely influential work of Henri Poincare on 3BP has end the classical period of work. King Oscar II of Sweden, in the late 19th century setup an award for solving the $n$-body problem on the recommendation of Karl Weierstrass, Gsta MittagLeffler, and Charles Hermite converges uniformly [2]. Many eminent mathematicians and astronomers like Carl Gustav Jacob Jacobi, Lagrange and Euler working on it in the 19th century. Until 1991, the general solution to the problem was remained unsolved, when a Professor in the University of Arizona, Qiudong Wang published "The global solution of $n$-body Problem" [3]. Gomatan et al. (1999), Kozak and Oniszk (1998) and Majorana (1981) derived equilibrium solutions and analyzed their stability for different types of four-body problems. Majorana (1981) studied the linear stability of the equilibrium points in the restricted four-body problem.More recent works on the collinear problem include those of Douskos (2010), and Ouyang and Xie (2005). Douskos discussed the existence and stability of the collinear equilibrium points of a generalized Hill problem and showed the existence of two equilibrium points for a positive oblateness co-efficient. Ouyang and Xie found regions on the configuration space where it is possible to choose masses for collinear configuration of four bodies which will make it central.

### 1.1 Central Configuration

A Central Configuration $(C C)$ is a special arrangement of point masses interacting by Newton's law of gravitation with the following property "The gravitational acceleration vector produced on each mass by all others should point toward the center of mass and proportional to the distance to the center

## of mass".

$C C$ play an important role in the study of the Newtonian $n$-body problem. For an arbitrary given set of masses the number of classes of planar non collinear central configurations of the $n$-body problem has been only solved for $n=3$. In this case they are the three collinear and the two equilateral triangle central configurations, due to Euler [4] and Lagrange. Recently, Hampton and Moeckel [5] proved that for any choice of four masses there exist a finite number of classes of central configurations. For five or more masses this result is unproved, but recently an important contribution to the case of five masses has been made by Albouy and Kaloshin [6]. Under the assumption that every central configuration of the four-body problem has an axis of symmetry when the four masses are equal, the central configurations were characterized studying the intersection points of two planar curves in [7]. Later on in [8, 9] Albouy provided a complete proof for the classes of central configurations of the four-body problem with equal masses. Bernat, Llibre and Perez Chavela [10] complete the characterization of the kite planar noncollinear classes of central configurations with three equal masses, started by Leandro in [11].

### 1.1.1 Restricted $n$-Body Problem

Restricted $n$ body problem is defined that $n-1$ masses $\left(m_{i}=3,4,5, \ldots, n-\right.$ 1) and one infinitesimal (test particle) mass $m$, which has negligible mass as compared with $m_{i}$, i.e., $m \ll m_{i}$. From the above defination we can easily conclude that the mass $m$ does not have any gravitaional influence on all $m_{i}$, due to the condition $m \ll m_{i}$. The first such problem restricted three body problem (R3TB) was described by Henri Pioncare [2]. Euler's solved the three-body problem for the motion of a particle that is influenced by the gravitational field of two other point masses fixed in space. This problem is explicitly solvable and provides an approximate solution for moving particles in the gravitational fields. A systematic analysis of periodic orbits was done in the problem of the two-dimensional, elliptic, restricted three-body [12]. The position and
stability of the five points of equilibrium in the planar, circular restricted threebody problem is investigated when a variety of studies of drag forces act on the third body [13].

In the restricted three-body problem, the presence of transversal ejection-collision orbits discussed [14]. Conley et al. discussed new long periodic solutions in plane, of the restricted three-body problem [15]. Lagrange points and their stability in a restricted four-body problem where three bodies are finite and fourth is infinitesimal, do not affect the movement of the three bodies moving in circles around their center of mass fixed at the origin explained in [16]. Sim-mons and Bakker gave analysis (linear stability) of a rhomboidal 4BP and show that collisions (isolated binary) can be regularized at origin [17]. Prokopenya discussed the stability of the equilibrium solutions in the elliptic restricted many-body problem [18]. Planar central configurations of the Four-Body Problem with three equal masses discussed in [10]. Santos discussed each equilibrium solution must be defined by the primaries along a diagonal [19].

### 1.2 Thesis Contribution

We are setting a restricted symmetric collinear six body problem that includes symmetrical arrangement of two pairs of masses and one mass is at origin. The masses are $m_{1}, m_{2}, m_{3}, m_{4}, m_{5}$ and $m_{6}$. Consider theses masses $m_{1}=m_{2}=m_{3}=$ $m, m_{4}=m_{5}=M$, and small mass $m_{6}$ moving so that their configuration is always in a straight line. Here we study the positions of equilibrium points of $m_{6}$ in the gravitational field of 5 big masses and we will check their stability.

### 1.3 Dissertation Outlines

We further divided this dissertation into 4 chapters.

## 1. Chapter 2:

This chapter includes some important definitions concerning the celestial mechanics, Newton's laws of motion and the planetary motion laws of Kepler, the two body problem (2BP) and solution of two body problem.

## 2. Chapter 3:

In this chapter characterization of collinear configuration is discussed.

## 3. Chapter 4:

In this chapter the dynamics of $6^{\text {th }}$ body, equilibrium solutions and analysis the stability of equilibrium points are briefly explained.

## 4. Chapter 5:

This chapter summarizes the whole study and includes the conclusion arising from entire discussion.

Bibliography contains a list of the references used in the dissertation.

## Chapter 2

## Some Preliminaries

This chapter contains fundamental definitions, fundamental concepts, universal principles and laws that will make our research work more comprehensible.
Definition 2.1.1. (Motion)
"Motion is the phenomenon in which an object changes its position over time. Motion is mathematically described in terms of displacement, distance, velocity, acceleration, speed, and time." [20]

## Definition 2.1.2. (Mechanics)

"Mechanics is the science that studied the motion of objects and can be divided into the following:

1. Kinematics, describes how objects move in terms of space and time.
2. Dynamics, described the cause of the object's motion.
3. Statics, deals with the conditions under which an object subjected to various forces is in equilibrium." [21]

## Definition 2.1.3. (Vectors)

"A vector quantity may be geometrically represented by a straight line, having a length proportional to the magnitude of the vector quantity and drawn in the
same direction and sense as that of the given vector quantity." [22]

## Definition 2.1.4. (Scalar)

"Many quantities in physics can be completely specified by giving their magnitude alone". [22]

## Definition 2.1.5. (Momentum)

"The linear momentum (or quantity of motion as was called by Newton) of a particle of mass $\boldsymbol{\rho}$ is a vector quantity defined as:

$$
\begin{equation*}
\boldsymbol{\rho}=m \mathbf{v}, \tag{2.1}
\end{equation*}
$$

where $\mathbf{v}$ is the velocity of the particle. A fast moving car has more momentum than a slow moving car of same mass." [23]

## Definition 2.1.6. (Conservation of Linear Momentum )

"If no net external force acts on a system of particles, the total linear momentum $\boldsymbol{\rho}$ of the system cannot change. This result is called the law of conservation of linear momentum. It can also be written as

$$
\rho_{i}=\rho_{f} .
$$

In other words, this equation says that, for a closed, isolated system, ( total linear momentum at some initial time $\left.t_{i}\right)=($ total linear momentum at some later time $t_{f}$ )." [23]

## Definition 2.1.7. (Newton's Second Law in Term of Momentum)

"Newton's second law can be expressed in terms of momentum for a particle like object of constant mass as

$$
\begin{gathered}
\mathbf{F}=m \mathbf{a}=\frac{d \mathbf{v}}{d t}=\frac{d}{d t}(m \mathbf{v})=\frac{d \boldsymbol{\rho}}{d t} . \\
\boldsymbol{\rho}=\sum_{1}^{n} \boldsymbol{\rho}_{i} .
\end{gathered}
$$

The total linear momentum $\boldsymbol{\rho}$ of a system of particles is defined as the vector sum of the individual linear momentum." [23]

## Definition 2.1.8. (Angular Momentum)

"Angular momentum $\mathbf{L}$ of a particle of mass $m$ and linear momentum $\mathbf{L}$ is a vector quantity defined as:

$$
\mathbf{L}=\mathbf{p} \times \rho
$$

where $\mathbf{p}$ is a position vector of a particle relative to an origin $O$ that is in an inertial frame." [23]

## Definition 2.1.9. (Conservation of Angular Momentum )

"Law of conservation of angular momentum, can also be written as or net angular momentum at some Initial time $\left.t_{i}\right)=($ net angular momentum at some later time $t_{f}$.) If the net external torque acting on a system is zero, the angular momentum $r$ of the system remains constant, no matter what changes take place within the system." [23]

## Definition 2.1.10. (Torque)

"A quantity called torque $\boldsymbol{\tau}$ as the product of the two factors and write it as

$$
\boldsymbol{\tau}=\mathbf{r} \times \mathbf{F}
$$

The magnitude of $\boldsymbol{\tau}$ is $\boldsymbol{\tau}=\mathrm{rF} \sin \boldsymbol{\theta}$, where r is the perpendicular distance between the rotation axis at 0 and an extended line running through the vector $\mathbf{F}$, and $\boldsymbol{\theta}$ is the angle between the position and force vectors." [23]

## Definition 2.1.11. (Central Force Field)

"A force is said to be central under two conditions. First, the direction of the force must always be towards or away from fixed point. The point is known as the center force. Second, the magnitude of the force should only proportional to the distance $r$ between the particle and center of the force. The central force may be
written as

$$
\begin{equation*}
\mathbf{F}=f(r) \mathbf{r}_{1} \tag{2.2}
\end{equation*}
$$

where $\mathbf{r}_{\mathbf{1}}$ is a unit vector in the direction of $\mathbf{r}$. The most widely known are the gravitational force and Coulomb force." [21]

## Definition 2.1.12. (Degree of Freedom)

"Consider the motion of free particle. To describe this motion we use three independent coordinates such as the Cartesian coordinates $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$. The particle is free to execute motion along any one axis independently with change in one coordinate only. The above statement is equivalent to saying that the particle has three degree of freedom." [22]

## Definition 2.1.13. (Center of Mass of System of Particle )

"Center of mass $\boldsymbol{c}$ of system of particle is the point that moves as through all of the system mass were concentrated there and all external forces were applied.For example, the center of mass of a uniform disc shape would be at its center. Sometimes the center of mass doesn't fall anywhere on the object. The center of mass of a ring for example is located at its center". [23]

$$
\begin{gathered}
\left(m_{1}+m_{2}+\ldots+m_{n}\right) \hat{\mathrm{r}}=m_{1} \mathrm{r}_{1}+m_{2} \mathrm{r}_{2}+\ldots+m_{n} \mathrm{r}_{n} \\
c=\frac{\left(m_{1}+m_{2}+\ldots+m_{n}\right) \hat{\mathrm{r}}}{M}
\end{gathered}
$$

where

$$
M=\sum_{i=1}^{n} \mathrm{~m}_{i}
$$

## Definition 2.1.14. (Center of Gravity)

"The gravitational force on an extended body is the vector sum of the gravitational forces acting on the individual elements (the atoms) of the body. Instead of considering all those individual elements, we can say that the gravitational force $\mathbf{F}_{\boldsymbol{g}}$ on a body effectively acts at a single point, called the center of gravity (cog)
of the body. An example of center of gravity is the middle of a seesaw." [23]

$$
G_{c}=\frac{W_{1} d_{1}+W_{2} d_{2}+W_{3} d_{3}+\ldots+W_{n} d_{n}}{W}
$$

## Definition 2.1.15. ( Principle of Superposition )

"This is a general principle that says a net effect is the sum of the individual effects." [23]

## Definition 2.1.16. ( Equilibrium )

"The two requirements for equilibrium
are:

1. The linear momentum $\boldsymbol{\rho}$ of its center of mass is constant.
2. Its angular momentum $\mathbf{L}$ about its center of mass, or about any other point, is also constant.

To find the zeros $(\boldsymbol{\xi}, \boldsymbol{\eta})$ or equilibrium points / Lagrange point, we need to solve th equations numerically or drawing contour plot using Mathematica. The classification of equilibrium points for restricted collinear six body problem is discussed.

We say that such objects are in equilibrium." [23]

## Definition 2.1.17. (Inertial Frame of Reference)

"A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called inertial frame of reference. Actually, an unaccelerated frame of reference is an inertial frame of reference. In this frame of reference a body does not acted upon by external forces. Newton's laws of motion are valid in all inertial frames of reference. All inertial frames of reference are equivalent." [23]
motion are valid in all inertial frames of reference. All inertial frames of reference are equivalent." [23]

## Definition 2.1.18. (Point-Like Particle)

"A point-like particle is an idealization of particles mostly used in different fields of physics. Its defining features is the lacks of spatial extension: being zerodimensional, it does not take up space. A point-like particle is an appropriate representation of an object whose structure, size and shape is irrelevant in a given context. e.g., from far away, a finite-size mass (object) will look like a point-like particle." [24]

## Definition 2.1.19. (Lagrange Points)

"The equilibrium solutions for the three-body problem are named after JosephLouis Lagrange, an 18th-century mathematician who wrote about them in 1772. A Lagrange point is a location in space where the combined gravitational forces of two large bodies, such as Earth and the sun or Earth and the moon, equal the gravitational force felt by a much smaller third body. Of the five Lagrange points,


Figure 2.1: Lagrane points
three are unstable and two are stable. The unstable Lagrange points - labeled $\boldsymbol{L}_{\mathbf{1}}, \boldsymbol{L}_{\mathbf{2}}$ and $\boldsymbol{L}_{\mathbf{3}}$ - lie along the line connecting the two large masses. The stable Lagrange points - labeled $\boldsymbol{L}_{\mathbf{4}}$ and $\boldsymbol{L}_{\mathbf{5}}$ - form the apex of two equilateral triangles
that have the large masses at their vertices. $\boldsymbol{L}_{\mathbf{4}}$ leads the orbit of earth and $\boldsymbol{L}_{\mathbf{5}}$ follows. The $\boldsymbol{L}_{\mathbf{1}}$ point of the Earth-Sun system affords an uninterrupted view of the sun and is currently home to the Solar and Heliospheric Observatory Satellite SOHO. The $\boldsymbol{L}_{\mathbf{2}}$ point of the Earth-Sun system was the home to the WMAP spacecraft, current home of Planck, and future home of the James Webb Space Telescope. $\boldsymbol{L}_{\mathbf{2}}$ is ideal for astronomy because a spacecraft is close enough to readily communicate with Earth, can keep Sun, Earth and Moon behind the spacecraft for solar power and (with appropriate shielding) provides a clear view of deep space for our telescopes. The $\boldsymbol{L}_{\mathbf{1}}$ and $\boldsymbol{L}_{\mathbf{2}}$ points are unstable on a time scale of approximately 23 days, which requires satellites orbiting these positions to undergo regular course and attitude corrections."

### 2.1 Kepler's Three Laws of Planetary Motion

"Kepler's three laws of planetary motion can be described as follows:

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit. Mathematically, Kepler's third law can be written as:

$$
T^{2}=\left(\frac{4 \pi^{2}}{G M_{s}}\right) r^{3}
$$

where $\boldsymbol{T}$ is the time period, $r$ is the semi major axis, Ms is the mass of sun and $G$ is the universal gravitational constant." [25]

### 2.2 Isaac Newton's Laws of Motion

"The following three laws of motion given by Newton are considered the axioms of mechanics:

## 1. First law of motion

Newton's first law of motion essentially states that a point object subject to zero net external force moves in a straight line with a constant speed (i.e., it does not accelerate). However, this is only true in special frames of reference called inertial frames. Indeed, we can think of Newton's first law as the definition of an inertial frame."

## 2. Second law of motion

"Newton's second law of motion essentially states that if a point object is subject to an external force $\mathbf{F}$, then its equation of motion is given by

$$
\mathrm{F}=\frac{d}{d t}(m \mathrm{v})=\frac{d \rho}{d t}
$$

If $m$ is independent of time this becomes

$$
\mathrm{F}=m \frac{d \mathrm{v}}{d t}=m \mathrm{a}
$$

where the momentum $\boldsymbol{\rho}$ is the product of the object's inertial mass $\boldsymbol{m}$ and its velocity v."

## 3. Third law of motion

"Consider a system of $\mathbf{N}$ mutually interacting point objects. Let the ith object, whose mass is $\boldsymbol{m}_{\boldsymbol{i}}$, be located at position vector $\mathbf{p}_{\boldsymbol{i}}$. Suppose that this object exerts a force $\mathbf{f}_{\boldsymbol{j} \boldsymbol{i}}$ on the $\boldsymbol{j} \boldsymbol{t h}$ object. Likewise, suppose that the $\boldsymbol{j} \boldsymbol{t h}$ object exerts a force $\mathbf{f}_{\boldsymbol{i} \boldsymbol{j}}$ on the $\boldsymbol{i t h}$ object. Newton's third law of motion essentially states that these two forces are equal and opposite, irrespective of their nature. In other words, $\mathrm{f}_{i j}=-\mathrm{f}_{j i}$.
For example, a book resting on a table applies a downward force equal to its weight on the table." [25]

### 2.2.1 Newton's Universal Law of Gravitation

"Newton's law of gravitation: Every particle attracts any other particle with a gravitational force of magnitude

$$
\mathrm{F}=G \frac{m_{1} m_{2}}{\mathrm{p}^{3}} \mathrm{p}
$$

Here $\boldsymbol{m}_{\mathbf{1}}$ and $\boldsymbol{m}_{\mathbf{2}}$ are the masses of the particles, $\mathbf{p}$ is the distance between them, and $\boldsymbol{G}$ is the gravitational constant, with a value that is now known to be $\boldsymbol{G}=$ $6.67 \times 10^{-11} \boldsymbol{m}^{\mathbf{3}} \mathrm{kg}^{-1} \mathrm{~s}^{-2}$ and F is the gravitational force acting on particle 1 $\left(\boldsymbol{m}_{\mathbf{1}}\right)$ due to particle $2\left(\boldsymbol{m}_{\mathbf{2}}\right)$. The force is directed toward particle 2 and is said to be an attractive force because particle 1 is attracted toward particle." [25]

### 2.3 The Two-Body Problem

"A two-body problem is a dynamical system that consists of two freely moving point objects exerting forces on one another. Assume the first object has a mass of $\boldsymbol{m}_{\mathbf{0}}$ and has theposition vector $\mathbf{p}_{\mathbf{1}}$. Similarly, the second object has a mass of $\boldsymbol{m}_{\mathbf{1}}$ and has theposition vector $\mathbf{p}_{\mathbf{2}}$. Let the force applied by first objecton the second is $\left(\mathbf{F}_{\mathbf{0 1}}\right)$ and, theforce exerted by second object on first is $\left(\mathbf{F}_{\mathbf{1 0}}\right)$. The corresponding equations are:

$$
\begin{gather*}
m_{1} \frac{d^{2} \mathrm{r}_{1}}{d t^{2}}=-\mathrm{F}_{10}  \tag{2.3}\\
m_{2} \frac{d^{2} \mathrm{r}_{2}}{d t^{2}}=\mathrm{F}_{01} "[25] \tag{2.4}
\end{gather*}
$$

### 2.3.1 The Two Body Problem Solution

"The governing law for the two-body is Newton's universal gravitational law:

$$
\begin{equation*}
\mathrm{F}=G \frac{m_{0} m_{1}}{\mathrm{p}^{3}} \mathrm{p} \tag{2.5}
\end{equation*}
$$

for two masses $\boldsymbol{m}_{\mathbf{0}}$ and $\boldsymbol{m}_{\boldsymbol{1}}$ separated by a distance of $\mathbf{p}$, and G is the universal gravitational constant. The aim here is to determine the path of the particles for any time $t$, if the initial positions and velocities are known.

In Figure 2.1, the force of attraction $\mathbf{F}_{\mathbf{0}}$ is directed along $\mathbf{p}$ towards $\boldsymbol{m}_{\mathbf{0}}$, while the force $\mathbf{F}_{\mathbf{1}}$ on $\boldsymbol{m}_{\mathbf{1}}$ is in opposite direction.

According to Newton's 3rd law of motion,

$$
\begin{equation*}
\mathrm{F}_{10}=-\mathrm{F}_{01} . \tag{2.6}
\end{equation*}
$$

From Figure 2.1,

$$
\begin{equation*}
\mathrm{F}_{10}=G \frac{m_{0} m_{1}}{\mathrm{p}^{3}} \mathrm{p} \tag{2.7}
\end{equation*}
$$

Using Newton's 2nd law of motion and by Equations (2.6) and (2.7), the equation of motion of the particles under the influence of their mutual gravitational attraction is

$$
\begin{equation*}
m_{0} \mathrm{p}_{1}^{\prime \prime}=m_{0} \frac{d^{2} \mathrm{p}_{1}}{d t^{2}}=G \frac{m_{0} m_{1}}{\mathrm{p}^{3}} \mathrm{p} \tag{2.8}
\end{equation*}
$$



Figure 2.2: Center of mass

$$
\begin{equation*}
m_{1} \mathrm{p}_{2}^{\prime \prime}=m_{1} \frac{d^{2} \mathrm{p}_{2}}{d t^{2}}=-G \frac{m_{0} m_{1}}{\mathrm{p}^{3}} \mathrm{p} \tag{2.9}
\end{equation*}
$$

where $O$ is the reference point and $\mathbf{p}_{\mathbf{1}}$ and $\mathbf{p}_{\mathbf{2}}$ are the position vectors of $\boldsymbol{m}_{\mathbf{0}}$ and $\boldsymbol{m}_{\boldsymbol{1}}$ respectively. Adding Equations (2.8) and (2.9), we get

$$
\begin{equation*}
m_{0} \mathrm{p}_{1}^{\prime \prime}+m_{1} \mathrm{p}_{2}^{\prime \prime}=0 \tag{2.10}
\end{equation*}
$$

Integrating the above equation:

$$
\begin{equation*}
m_{0} \mathrm{p}_{1}^{\prime}+m_{1} \mathrm{p}_{2}^{\prime}=\mathrm{k}_{1} \tag{2.11}
\end{equation*}
$$

that the total linear momentum of the system i.e., $\boldsymbol{m}_{\mathbf{0}} \mathbf{v}_{\boldsymbol{m}_{0}}+\boldsymbol{m}_{\mathbf{1}} \mathbf{v}_{\boldsymbol{m}_{1}}=\mathrm{k}_{\mathbf{1}}$.
Again integrate Equation (2.11):

$$
\begin{equation*}
m_{0} \mathrm{p}_{1}+m_{1} \mathrm{p}_{2}=\mathrm{k}_{1} t+\mathrm{k}_{2} \tag{2.12}
\end{equation*}
$$

where $\mathbf{k}_{\mathbf{1}}$ and $\mathbf{k}_{\mathbf{2}}$ are constant vectors.
Using the concept of center of mass from $2 \mathrm{BP}, \mathbf{P}$ is defined as:

$$
\begin{align*}
& \mathrm{P}=\frac{m_{0} \mathrm{p}_{1}+m_{1} \mathrm{p}_{2}}{\left(m_{0}+m_{1}\right)},  \tag{2.13}\\
& \mathrm{P}=\frac{m_{0} \mathrm{p}_{1}+m_{1} \mathrm{p}_{2}}{m_{t}}, \tag{2.14}
\end{align*}
$$

where $\boldsymbol{m}_{\boldsymbol{t}}=\boldsymbol{m}_{\mathbf{0}}+\boldsymbol{m}_{\mathbf{1}}$.
When we compare the derivative of Equation (2.14) with Equation (2.11), we obtain

$$
\begin{gathered}
m_{t} \mathrm{P}^{\prime}=\mathrm{k}_{1} \\
\Rightarrow \quad \mathrm{P}^{\prime}=\frac{\mathrm{k}_{1}}{m_{1 t}}=\text { constant } .
\end{gathered}
$$

$\mathbf{P}^{\prime}=\mathbf{v}_{\boldsymbol{c}}$ is constant, in which the velocity of the center of mass is $\mathbf{v}_{\boldsymbol{c}}$.
Subtracting Equations (2.8) and (2.9), we get:

$$
\begin{align*}
& \mathrm{p}_{1}^{\prime \prime}-\mathrm{p}_{2}^{\prime \prime}=\frac{G m_{1}}{\mathrm{p}^{3}} \mathrm{p}+\frac{G m_{0}}{\mathrm{p}^{3}} \mathrm{p}  \tag{2.15}\\
& \mathrm{p}_{1}^{\prime \prime}-\mathrm{p}_{2}^{\prime \prime}=G \frac{\mathrm{p}}{\mathrm{p}^{3}}\left(m_{0}+m_{1}\right) \\
& \Rightarrow \mathrm{p}^{\prime \prime}=\alpha\left(\frac{\mathrm{p}}{\mathrm{p}^{3}}\right) \\
& \Rightarrow \mathrm{p}^{\prime \prime}+\alpha \frac{\mathrm{p}}{\mathrm{p}^{3}}=0 \tag{2.16}
\end{align*}
$$

where the reduced mass is defined as $\boldsymbol{\alpha}=\boldsymbol{G}\left(\boldsymbol{m}_{\mathbf{0}}+\boldsymbol{m}_{\mathbf{1}}\right)$ and $\mathrm{p}_{\mathbf{1}}-\mathrm{p}_{\mathbf{2}}=-\mathrm{p}$, see Figure 2.1.

With $\mathbf{p}$ as the cross product of Equation (2.16), we get:

$$
\mathrm{p} \times \alpha \mathrm{p}^{\prime \prime}+\frac{\alpha^{2}}{\mathrm{p}^{3}} \mathrm{p} \times \mathrm{p}=0
$$

$$
\begin{equation*}
\Rightarrow \mathrm{p} \times \mathrm{p}^{\prime \prime}=0 \tag{2.17}
\end{equation*}
$$

Integrating above equation:

$$
\begin{equation*}
\mathbf{p} \times \mathbf{p}^{\prime}=\mathbf{L} \tag{2.18}
\end{equation*}
$$

where $\mathbf{L}$ is a constant vector (Angular momentum), (2.18) may be expressed as,

$$
\begin{gathered}
\Rightarrow \mathrm{p} \times \alpha \mathrm{p}^{\prime \prime}=0 \\
\Rightarrow \mathrm{p} \times \mathrm{F}=0
\end{gathered}
$$

where $\mathbf{F}=\boldsymbol{\alpha} \mathbf{p}$ " $=\boldsymbol{\alpha} \mathbf{a}(\boldsymbol{\alpha})$ is defined as "reduced mass" or "constant".

According to definition of the angular momentum and torque, we obtain:

$$
\begin{equation*}
\tau^{*}=\frac{d \mathrm{~K}}{d t}=\mathrm{p} \times \mathrm{F} \tag{2.19}
\end{equation*}
$$

The result of comparing Equations (2.21) and (2.22) is:

$$
\begin{gathered}
\tau^{*}=\frac{d \mathrm{~K}}{d t} \\
\mathrm{p} \times \mathrm{F}=0 \\
\frac{d \mathrm{~K}}{d t}=0 \\
\Rightarrow \mathrm{~K}=\text { constant },
\end{gathered}
$$

This demonstrates that the system's angular momentum is constant." [25]

### 2.3.2 The Transverse Components and Radial Components of Velocity and Acceleration:

If $\mathbf{p}$ and $\boldsymbol{\theta}$ are the polar coordinates in this plane shown in Figure 2.2. $\mathbf{p}^{\prime}$ and $\mathbf{p} \boldsymbol{\theta}^{\prime}$ are the velocity components along and perpendicular to the radius vector
connecting $\boldsymbol{m}_{\mathbf{0}}$ and $\boldsymbol{m}_{\mathbf{1}}$, then,

$$
\begin{equation*}
\mathrm{p}^{\prime}=\frac{d \mathrm{p}}{d t}=\mathrm{p}^{\prime} \hat{i}+\mathrm{p} \theta^{\prime} \hat{j} \tag{2.20}
\end{equation*}
$$

where $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are unit vectors parallel and perpendicular to the radius vector. As a result of Equations (2.19) and (2.24),

$$
\begin{equation*}
\mathrm{p} \times\left(\mathrm{p}^{\prime} \hat{\boldsymbol{i}}+\mathrm{p} \boldsymbol{\theta}^{\prime} \hat{\boldsymbol{j}}\right)=\mathrm{p}^{2} \boldsymbol{\theta}^{\prime} \hat{\boldsymbol{k}}=\boldsymbol{K} \hat{\boldsymbol{k}} \tag{2.21}
\end{equation*}
$$

where $\hat{\boldsymbol{k}}$ is a unit vector perpendicular to the orbit's plane. We are able to write

$$
\begin{equation*}
\mathrm{p}^{2} \theta^{\prime}=K \tag{2.22}
\end{equation*}
$$

thus the constant $\boldsymbol{K}$ is seen to be twice the rate of description of area by the radius vector, which is the mathematical expression of Kepler's second law.


Figure 2.3: The transverse components and radial components of velocity and acceleration

If we take the scalar product of Equation (2.16) with $\mathbf{p}^{\prime}$, we obtain :

$$
\mathrm{p}^{\prime} \cdot \frac{d^{2} \mathrm{p}}{d t^{2}}+\alpha \frac{\mathrm{p}^{\prime} \cdot \mathrm{p}}{\mathrm{p}^{3}}=0
$$

by integrate the above equation, we get

$$
\begin{gather*}
\frac{1}{2} \mathrm{p}^{\prime} \cdot \mathrm{p}^{\prime}-\frac{m_{0} u}{\mathrm{p}}=C  \tag{2.23}\\
\frac{1}{2} v^{2}-\frac{\alpha}{\mathrm{p}}=C \tag{2.24}
\end{gather*}
$$

where $C$ is a constant.
Components of the acceleration vector are paralleland perpendicular to the radius vector, as determined by celestial mechanics (see Figure 2.2):

$$
\mathrm{b}=\left(\mathrm{p}^{\prime \prime}-\mathrm{p} \theta^{\prime 2}\right) \hat{i}+\frac{1}{\mathrm{p}} \frac{d}{d t}\left(\mathrm{p}^{2} \theta^{\prime}\right) \hat{j}
$$

using this Equation in (2.11), we get

$$
\begin{gather*}
\mathrm{p}^{\prime \prime}-\mathrm{p} \theta^{\prime 2}=-\frac{\alpha}{\mathrm{p}^{2}}  \tag{2.25}\\
\frac{1}{\mathrm{p}} \frac{d}{d t}\left(\mathrm{p}^{2} \theta^{\prime}\right)=0 \tag{2.26}
\end{gather*}
$$

Integrating Equation (2.30):

$$
\begin{equation*}
\mathrm{p}^{2} \theta^{\prime}=K \tag{2.27}
\end{equation*}
$$

which is the angular momentum integral, using the standard replacement of

$$
\begin{equation*}
w=\frac{1}{\mathrm{p}} \tag{2.28}
\end{equation*}
$$

removing the time between (2.20) and (2.22) Equations:

$$
\begin{align*}
\Rightarrow & \frac{d^{2} w}{d \theta^{2}}+w=\frac{\alpha}{K^{2}}  \tag{2.29}\\
\Rightarrow w & =\frac{\alpha}{K^{2}}+B \cos \left(\theta-\theta_{0}\right) \tag{2.30}
\end{align*}
$$

Substitute $\boldsymbol{w}=\frac{1}{\mathrm{p}}$ in above equation:

$$
\frac{1}{\mathrm{p}}=\frac{\alpha}{K^{2}}+B \cos \left(\theta-\theta_{0}\right)
$$

$$
\Rightarrow \mathrm{p}=\frac{\frac{K^{2}}{\alpha}}{1+\frac{K^{2} B}{\alpha} \cos \left(\theta-\theta_{0}\right)}
$$

is the polar form of the equation of the conic which may be written as:

$$
\mathrm{p}=\frac{q}{1+e^{*} \cos \left(\theta-\theta_{0}\right)}
$$

where

$$
\begin{aligned}
q & =\frac{K^{2}}{\alpha} \\
e^{*} & =\frac{B K^{2}}{\alpha}
\end{aligned}
$$

Eccentricity $e^{*}$ classifies the trajectory of one celestial body around another.

## 1. Elliptical orbit:

An elliptical orbit is one in which $0<e^{*}<\mathbf{1}$.

## 2. Parabolic orbit:

Parabolic orbit occurs when $\boldsymbol{e}^{*}=1$.

## 3. Hyperbolic orbit:

In the case of $e^{*}>1$, the orbit is a hyperbolic orbital trajectory.

As a result, the solution to the two-body problem (2BP) is a conic section with Kepler's first law as a particular case.

## Chapter 3

## Restricted Symmetric Collinear Central Configuration for Six Body

### 3.1 Introduction

A collinear five body problem that includes symmetrical arrangement of two pairs of equal masses along each side of center of mass and one mass at the origin is investigated by using Newton's laws of motion and universal law of gravity. The masses are $\boldsymbol{m}_{\mathbf{1}}, \boldsymbol{m}_{\mathbf{2}}, \boldsymbol{m}_{\mathbf{3}}, \boldsymbol{m}_{\mathbf{4}}$ and $\boldsymbol{m}_{\mathbf{5}}$. The central configuration is analyzed with the assumption $\boldsymbol{m}_{1}=\boldsymbol{m}_{\mathbf{2}}=\boldsymbol{m}_{\mathbf{3}}=\boldsymbol{m}$ and $\boldsymbol{m}_{\mathbf{4}}=\boldsymbol{m}_{\mathbf{5}}=\boldsymbol{M}$. The larger masses are placed in the middle and smaller masses are placed at the corner of each side.

### 3.2 Collinear Configuration Characterization

For $\boldsymbol{n}$-body problem the classical equation of motion is

$$
\begin{equation*}
\sum_{k=1}^{n} m_{k \neq j} \frac{\mathbf{p}_{k}-\mathbf{p}_{j}}{\left|\mathbf{p}_{k}-\mathbf{p}_{j}\right|^{3}}=\mathrm{p}_{j}^{\prime \prime} . \tag{3.1}
\end{equation*}
$$

A central configuration of the $\boldsymbol{n}$ bodies where the acceleration vector of each body is proportional to its position vector, and the constant of proportionality is the same for $\boldsymbol{n}$ bodies. Therefore, a central configuration fulfills the equation:

$$
\begin{equation*}
\sum_{k=1 k \neq j}^{n} m_{k} \frac{\left(\mathrm{p}_{k}-\mathrm{p}_{j}\right)}{\left|\mathrm{p}_{k}-\mathrm{p}_{j}\right|^{3}}=-\omega^{2}\left(\mathrm{p}_{j}-\mathrm{c}\right) \tag{3.2}
\end{equation*}
$$

here $\boldsymbol{\omega}^{\mathbf{2}}$ is an angular velocity which is also nonzero as well as constant, where $\mathbf{c}$ is defined as

$$
\begin{equation*}
\mathbf{c}=\frac{\sum_{j=1}^{n} \boldsymbol{m}_{j} \mathbf{p}_{j}}{\sum_{j=1}^{n} \boldsymbol{m}_{j}} \tag{3.3}
\end{equation*}
$$

wich is the center of mass for $n$ bodies.

Substitute $\boldsymbol{n}=5$ in equation number (3.2), the CC's equations for the general five-body problem (5BP) are:
$m_{2} \frac{\left(\mathrm{p}_{2}-\mathrm{p}_{1}\right)}{\left|\mathrm{p}_{2}-\mathrm{p}_{1}\right|^{3}}+m_{3} \frac{\left(\mathrm{p}_{3}-\mathrm{p}_{1}\right)}{\left|\mathrm{p}_{3}-\mathrm{p}_{1}\right|^{3}}+m_{4} \frac{\left(\mathrm{p}_{4}-\mathrm{p}_{1}\right)}{\left|\mathrm{p}_{4}-\mathrm{p}_{1}\right|^{3}} m_{5} \frac{\left(\mathrm{p}_{5}-\mathrm{p}_{1}\right)}{\left|\mathrm{p}_{5}-\mathrm{p}_{1}\right|^{3}}=-\omega^{2}\left(\mathrm{p}_{1}-\mathrm{c}\right)$,
$m_{1} \frac{\left(\mathrm{p}_{1}-\mathrm{p}_{2}\right)}{\left|\mathrm{p}_{1}-\mathrm{p}_{2}\right|^{3}}+m_{3} \frac{\left(\mathrm{p}_{3}-\mathrm{p}_{2}\right)}{\left|\mathrm{p}_{3}-\mathrm{p}_{2}\right|^{3}}+m_{4} \frac{\left(\mathrm{p}_{4}-\mathrm{p}_{2}\right)}{\left|\mathrm{p}_{4}-\mathrm{p}_{2}\right|^{3}}+m_{5} \frac{\left(\mathrm{p}_{5}-\mathrm{p}_{2}\right)}{\left|\mathrm{p}_{5}-\mathrm{p}_{2}\right|^{3}}=-\omega^{2}\left(\mathrm{p}_{2}-\mathrm{c}\right)$,
$m_{1} \frac{\left(\mathrm{p}_{1}-\mathrm{p}_{3}\right)}{\left|\mathrm{p}_{1}-\mathrm{p}_{3}\right|^{3}}+m_{2} \frac{\left(\mathrm{p}_{2}-\mathrm{p}_{3}\right)}{\left|\mathrm{p}_{2}-\mathrm{p}_{3}\right|^{3}}+m_{4} \frac{\left(\mathrm{p}_{4}-\mathrm{p}_{3}\right)}{\left|\mathrm{p}_{4}-\mathrm{p}_{3}\right|^{3}}+m_{5} \frac{\left(\mathrm{p}_{5}-\mathrm{p}_{3}\right)}{\left|\mathrm{p}_{5}-\mathrm{p}_{3}\right|^{3}}=-\omega^{2}\left(\mathrm{p}_{3}-\mathrm{c}\right)$,
$m_{1} \frac{\left(\mathrm{p}_{1}-\mathrm{p}_{4}\right)}{\left|\mathrm{p}_{1}-\mathrm{p}_{4}\right|^{3}}+m_{2} \frac{\left(\mathrm{p}_{2}-\mathrm{p}_{4}\right)}{\left|\mathrm{p}_{2}-\mathrm{p}_{4}\right|^{3}}+m_{3} \frac{\left(\mathrm{p}_{3}-\mathrm{p}_{4}\right)}{\left|\mathrm{p}_{3}-\mathrm{p}_{4}\right|^{3}}+m_{5} \frac{\left(\mathrm{p}_{5}-\mathrm{p}_{4}\right)}{\left|\mathrm{p}_{5}-\mathrm{p}_{4}\right|^{3}}=-\omega^{2}\left(\mathrm{p}_{4}-\mathrm{c}\right)$,
$m_{1} \frac{\left(\mathrm{p}_{1}-\mathrm{p}_{5}\right)}{\left|\mathrm{p}_{1}-\mathrm{p}_{5}\right|^{3}}+m_{2} \frac{\left(\mathrm{p}_{2}-\mathrm{p}_{5}\right)}{\left|\mathrm{p}_{2}-\mathrm{p}_{5}\right|^{3}}+m_{3} \frac{\left(\mathrm{p}_{3}-\mathrm{p}_{5}\right)}{\left|\mathrm{p}_{3}-\mathrm{p}_{5}\right|^{3}}+m_{5} \frac{\left(\mathrm{p}_{4}-\mathrm{p}_{5}\right)}{\left|\mathrm{p}_{4}-\mathrm{p}_{5}\right|^{3}}=-\omega^{2}\left(\mathrm{p}_{5}-\mathrm{c}\right)$.

Now consider the five masses $\boldsymbol{m}_{\mathbf{1}}, \boldsymbol{m}_{\mathbf{2}}, \boldsymbol{m}_{\mathbf{3}}, \boldsymbol{m}_{\mathbf{4}}, \boldsymbol{m}_{\mathbf{5}}$ with their positions at $\mathbf{p}_{\mathbf{1}}=$ $(0,0), \mathrm{p}_{2}=(-b, 0), \mathrm{p}_{3}=(b, 0), \mathrm{p}_{4}=(-a, 0), \mathrm{p}_{5}=(a, 0)$ respectively. Assuming the masses as:

$$
\begin{equation*}
m_{1}=m_{2}=m_{3}=m \quad \text { and } \quad m_{4}=m_{5}=M \tag{3.9}
\end{equation*}
$$

and the distances between the masses due to the symmetry of the geometry (see Figure 3.1) satisfy the following relations:

$$
\begin{equation*}
\mathrm{p}_{2}=-\mathrm{p}_{3}, \quad \mathrm{p}_{4}=-\mathrm{p}_{5} \tag{3.10}
\end{equation*}
$$



Figure 3.1: Symmetric collinear central configuration for five body

The center of mass for five-bodies can be written as,

$$
\begin{equation*}
\mathrm{c}=\frac{m_{1} \mathrm{p}_{1}+m_{2} \mathrm{p}_{2}+m_{3} \mathrm{p}_{3}+m_{4} \mathrm{p}_{4}+m_{5} \mathrm{p}_{5}}{m_{1}+m_{2}+m_{3}+m_{4}+m_{5}} \tag{3.11}
\end{equation*}
$$

after using the values of $\mathbf{p}_{1}, \mathbf{p}_{2}, \mathrm{p}_{3}, \mathrm{p}_{4}, \mathbf{p}_{5}$ and $\boldsymbol{m}_{1}=\boldsymbol{m}_{2}=\boldsymbol{m}_{3}=\boldsymbol{m}$ and $\boldsymbol{m}_{\mathbf{4}}=\boldsymbol{m}_{\mathbf{5}}=\boldsymbol{M}$, the equation (3.11) becomes:

$$
\begin{equation*}
\mathrm{c}=(0,0) \tag{3.12}
\end{equation*}
$$

Now using the values of Equations (3.9), (3.11) and (3.12) in Equations (3.4)-(3.8) and using $\boldsymbol{\omega}^{\mathbf{2}}=\mathbf{1}$ (without loss of generality), then the equation (3.4) is identically satisfied and remaining four Equations (3.5)-(3.8) become:

$$
\begin{gather*}
\frac{M}{(a+b)^{2}}+\frac{5 m}{4 b^{2}}+\frac{M}{(a-b)^{2}}-b=0  \tag{3.13}\\
\frac{M}{(a-b)^{2}}-\frac{5 m}{4 b^{2}}+\frac{M}{(a+b)^{2}}+b=0  \tag{3.14}\\
\frac{m}{(a+b)^{2}}+\frac{m}{(a-b)^{2}}+\frac{m}{a^{2}}+\frac{M}{4 a^{2}}-a=0  \tag{3.15}\\
\frac{m}{(a-b)^{2}}+\frac{m}{(a+b)^{2}}-\frac{m}{a^{2}}-\frac{M}{4 a^{2}}+a=0 \tag{3.16}
\end{gather*}
$$

Equation (3.13) and (3.15) are similar to Equation (3.14) and (3.16) respectively. Therefore we have the following two equations:

$$
\begin{gather*}
\frac{M}{(a-b)^{2}}-\frac{5 m}{4 b^{2}}+\frac{M}{(a+b)^{2}}+b=0  \tag{3.17}\\
\frac{m}{(a+b)^{2}}+\frac{m}{(a-b)^{2}}+\frac{m}{a^{2}}+\frac{M}{4 a^{2}}-a=0 \tag{3.18}
\end{gather*}
$$

Solving the above equations for $\boldsymbol{m}$ and $\boldsymbol{M}$, we get

$$
\begin{equation*}
m=\frac{h_{1}(a, b)}{h_{2}(a, b)} \tag{3.19}
\end{equation*}
$$

where

$$
\begin{gather*}
h_{1}(a, b)=-\left(\frac{b}{4 a^{2}}+\frac{a}{(a-b)^{2}}+\frac{a}{(a+b)^{2}}\right)  \tag{3.20}\\
h_{2}(a, b)=-\frac{5}{16 a^{2} b^{2}}-\left(\frac{1}{(a-b)^{2}}+\frac{1}{(a+b)^{2}}\right)\left(\frac{1}{a^{2}}+\frac{1}{(a-b)^{2}}+\frac{1}{(a+b)^{2}}\right), \tag{3.21}
\end{gather*}
$$

and

$$
\begin{equation*}
M=\frac{h_{3}(a, b)}{h_{4}(a, b)}, \tag{3.22}
\end{equation*}
$$

where

$$
\begin{equation*}
h_{3}(a, b)=b+4 a^{2}+4 a^{3}\left(\frac{1}{(a-b)^{2}}+\frac{1}{(a+b)^{2}}\right)\left(\frac{1}{a^{2}}+\frac{1}{(a-b)^{2}}+\frac{1}{(a+b)^{2}}\right) \tag{3.23}
\end{equation*}
$$

$$
\begin{equation*}
h_{4}(a, b)=-\frac{5}{16 a^{2} b^{2}}-\left(\frac{1}{(a-b)^{2}}+\frac{1}{(a+b)^{2}}\right)\left(\frac{1}{a^{2}}+\frac{1}{(a-b)^{2}}+\frac{1}{(a+b)^{2}}\right) \tag{3.24}
\end{equation*}
$$

Our next objective is to verify the positivity of small mass $\boldsymbol{m}$ and the large $\boldsymbol{M}$, which were described in equation (3.19) and (3.22) i.e., to find the value of $\boldsymbol{a}$ and the value of $\boldsymbol{b}$ for which the $\boldsymbol{m}$ and $\boldsymbol{M}$ masses are positive. Equations (3.19) and (3.22) are non-linear algebraic equations. It is very difficult to solve these equations for $\boldsymbol{m}$ and $\boldsymbol{M}$. Taking $\boldsymbol{b}=\mathbf{0 . 5}$ and solving (3.19) and (3.22) for $\boldsymbol{a}$, we get the following two cases for $\boldsymbol{a}$, where $\boldsymbol{m}$ and $\boldsymbol{M}$ are positive.i.e.,

1. $0.20 \leq a \leq 0.250$
2. $a \geq 0.87$

## Chapter 4

## Dynamics of $6^{\text {th }}$ Particle

### 4.1 Introduction

The dynamics of $\boldsymbol{6}^{\boldsymbol{t h}}$ particle $\boldsymbol{m}_{\mathbf{6}}$ moving in the plane according to the gravitational field formed by the attraction of $\mathbf{5}$ masses $\left(\boldsymbol{m}_{\mathbf{1}}, \boldsymbol{m}_{\mathbf{2}}, \boldsymbol{m}_{\mathbf{3}}, \boldsymbol{m}_{\mathbf{4}}, \boldsymbol{m}_{\mathbf{5}}\right)$ moving always in a straight line is discussed. The motion of $\boldsymbol{m}_{\mathbf{6}}$ will not effect the gravitational field of $m_{1}, m_{2}, m_{3}, m_{4}$ and $m_{5}$ because $\boldsymbol{m}_{6} \ll m_{1}, m_{2}, m_{3}, m_{4}, m_{5}$. This problem is called restricted collinear six-body problem (RC6BP). Equilibrium points, their stabilities and regions of permissible motions of $\boldsymbol{m}_{\mathbf{6}}$ according to the Jacobian constant is also investigated. Equation of motion that describe the planer motion of restricted $\boldsymbol{6}^{\boldsymbol{t h}}$ particle, mass $\boldsymbol{m}_{\mathbf{6}}$ written from equation (3.1) reads in inertial frame of reference, as:

$$
\begin{align*}
\mathrm{p}_{6}^{\prime \prime}=m_{5} \frac{\mathrm{p}_{5}-\mathrm{p}_{6}}{\left|\mathrm{p}_{5}-\mathrm{p}_{6}\right|^{3}} & +m_{4} \frac{\mathrm{p}_{4}-\mathrm{p}_{6}}{\left|\mathrm{p}_{4}-\mathrm{p}_{6}\right|^{3}}+m_{3} \frac{\mathrm{p}_{3}-\mathrm{p}_{6}}{\left|\mathrm{p}_{3}-\mathrm{p}_{6}\right|^{3}} \\
& +m_{2} \frac{\mathrm{p}_{2}-\mathrm{p}_{6}}{\left|\mathrm{p}_{2}-\mathrm{p}_{6}\right|^{3}}+m_{1} \frac{\mathrm{p}_{1}-\mathrm{p}_{6}}{\left|\mathrm{p}_{1}-\mathrm{p}_{6}\right|^{3}} \tag{4.1}
\end{align*}
$$

### 4.2 Equation of Motion of $\boldsymbol{m}_{6}$

Now our next goal is to write the above equation in rotating frame. Consider a coordinate system that is rotating about the center of mass with uniform angular
speed $\boldsymbol{\omega}$. Let $(\boldsymbol{\xi}, \boldsymbol{\eta})$ be the coordinates of $\boldsymbol{m}_{\mathbf{6}}$ in this new rotating frame (noninertial frame). The position vector of $\boldsymbol{m}_{\mathbf{6}}$ in the rotating frame is

$$
\begin{equation*}
\mathrm{p}_{6}=\xi(t) \mathrm{e}_{1}^{*}+\eta(t) \mathrm{e}_{2}^{*} \tag{4.2}
\end{equation*}
$$

where

$$
\mathrm{e}_{1}^{*}=e^{i w t}, \quad \mathrm{e}_{2}^{*}=i e^{i w t}
$$

Taking $\mathbf{1}^{\text {st }}$ derivative and $\mathbf{2}^{\text {nd }}$ derivatives of equation (4.2) and choose $\boldsymbol{\omega}^{\boldsymbol{2}}=\mathbf{1}$ we get the following expression for velocity and acceleration for $\boldsymbol{m}_{\mathbf{6}}$ in rotating frame.

$$
\left.\begin{array}{r}
\mathrm{p}_{6}^{\prime}=\left(\xi^{\prime}-\eta\right) e^{i t}+i\left(\xi+\eta^{\prime}\right) e^{i t}  \tag{4.3}\\
\mathrm{p}_{6}^{\prime \prime}=\left(\xi^{\prime \prime}-2 \eta^{\prime}-\xi\right) e^{i t}+i\left(\eta^{\prime \prime}+2 \xi^{\prime}-\eta\right) e^{i t}
\end{array}\right\} .
$$

Using Equation (4.3) in Equation (4.1), the planer equations of motion of $\boldsymbol{m}_{\mathbf{6}}$ in rotating frame in component form are

$$
\begin{equation*}
\xi^{\prime \prime}-2 \eta^{\prime}-\xi=-\left[m\left(\frac{\xi}{\mathrm{p}_{61}^{3}}+\frac{\xi+b}{\mathrm{p}_{62}^{3}}+\frac{\xi-b}{\mathrm{p}_{63}^{3}}\right)+M\left(\frac{\xi+a}{\mathrm{p}_{64}^{3}}+\frac{\xi-a}{\mathrm{p}_{65}^{3}}\right)\right] \tag{4.4}
\end{equation*}
$$

$$
\begin{equation*}
\eta^{\prime \prime}+2 \xi^{\prime \prime}-\eta=-\left[m\left(\frac{\eta}{\mathrm{p}_{61}^{3}}+\frac{\eta}{\mathrm{p}_{62}^{3}}+\frac{\eta}{\mathrm{p}_{63}^{3}}\right)+M\left(\frac{\eta}{\mathrm{p}_{64}^{3}}+\frac{\eta}{\mathrm{p}_{65}^{3}}\right)\right] \tag{4.5}
\end{equation*}
$$

The mutual distances are described as,

$$
\left.\begin{array}{r}
\mathrm{p}_{61}=\sqrt{\xi^{2}+\eta^{2}},  \tag{4.6}\\
\mathrm{p}_{62}=\sqrt{(\xi+b)^{2}+\eta^{2}}, \\
\mathrm{p}_{63}=\sqrt{(\xi-b)^{2}+\eta^{2}}, \\
\mathrm{p}_{64}=\sqrt{(\xi+a)^{2}+\eta^{2}}, \\
\mathrm{p}_{65}=\sqrt{(\xi-a)^{2}+\eta^{2}} .
\end{array}\right\}
$$

Multiplying Equation (4.4) by $\boldsymbol{\xi}^{\prime}$, and Equation (4.5) by $\boldsymbol{\eta}^{\prime}$, we get

$$
\begin{align*}
\xi^{\prime \prime} \xi^{\prime}-2 \eta^{\prime} \xi^{\prime}-\xi \xi^{\prime}= & -m \xi^{\prime}\left(\frac{\xi}{\mathrm{p}_{61}^{3}}+\frac{\xi+b}{\mathrm{p}_{62}^{3}}+\frac{\xi-b}{\mathrm{p}_{63}^{3}}\right) \\
& -M \xi^{\prime}\left(\frac{\xi+a}{\mathrm{p}_{64}^{3}}+\frac{\xi-a}{\mathrm{p}_{65}^{3}}\right),  \tag{4.7}\\
\eta^{\prime \prime} \eta^{\prime}+2 \xi^{\prime} \eta^{\prime}-\eta \eta^{\prime}= & -m \eta^{\prime}\left(\frac{\eta}{\mathrm{p}_{61}^{3}}+\frac{\eta}{\mathrm{p}_{62}^{3}}+\frac{\eta}{\mathrm{p}_{63}^{3}}\right) \\
& -M \eta^{\prime}\left(\frac{\eta}{\mathrm{p}_{64}^{3}}+\frac{\eta}{\mathrm{p}_{65}^{3}}\right) . \tag{4.8}
\end{align*}
$$

Adding Equations (4.7) and (4.8), we get we get:

$$
\begin{align*}
\xi^{\prime \prime} \xi^{\prime}+\eta^{\prime \prime} \eta^{\prime}-\left(\xi \xi^{\prime}+\eta \eta^{\prime}\right)= & -\frac{m}{\mathrm{p}_{61}^{3}}\left(\xi \xi^{\prime}+\eta \eta^{\prime}\right) \\
& -\frac{m}{\mathrm{p}_{62}^{3}}\left(\xi \xi^{\prime}+b \xi^{\prime}+\eta \eta^{\prime}\right) \\
& -\frac{m}{\mathrm{p}_{63}^{3}}\left(\xi \xi^{\prime}-b \xi^{\prime}+\eta \eta^{\prime}\right) \\
& -\frac{M}{\mathrm{p}_{64}^{3}}\left(\xi \xi^{\prime}+a \xi^{\prime}+\eta \eta^{\prime}\right) \\
& -\frac{M}{\mathrm{p}_{65}^{3}}\left(\xi \xi^{\prime}-a \xi^{\prime}+\eta \eta^{\prime}\right) . \tag{4.9}
\end{align*}
$$

Note that

$$
\begin{equation*}
\xi^{\prime \prime} \xi^{\prime}+\eta^{\prime \prime} \eta^{\prime}=\frac{1}{2} \frac{d}{d t}\left(\xi^{\prime 2}+\eta^{\prime 2}\right)=\frac{1}{2} \frac{d v^{2}}{d t} \tag{4.10}
\end{equation*}
$$

where $\boldsymbol{v}$ is the speed of the infinitesimal mass relative to the rotating frame. Similarly

$$
\begin{equation*}
\xi \xi^{\prime}+\eta \eta^{\prime}=\frac{1}{2} \frac{d}{d t}\left(\xi^{2}+\eta^{2}\right) \tag{4.11}
\end{equation*}
$$

From Equation (4.6), we obtaion, the following

$$
\begin{gather*}
\mathrm{p}_{61}^{2}=\xi^{2}+\eta^{2}  \tag{4.12}\\
\frac{d}{d t} \mathrm{p}_{61}=\frac{1}{\mathrm{p}_{61}}\left(\xi \xi^{\prime}+\eta \eta^{\prime}\right) . \tag{4.13}
\end{gather*}
$$

As we know that

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{1}{\mathrm{p}_{61}}\right)=-\frac{1}{\mathrm{p}_{61}^{2}} \frac{d}{d t} \mathrm{p}_{61} . \tag{4.14}
\end{equation*}
$$

Using Equation (4.13) in Equation (4.14) we get the following equations:

$$
\begin{equation*}
\frac{1}{\mathrm{p}_{61}^{\prime}}=-\frac{1}{\mathrm{p}_{61}^{3}}\left(\xi \xi^{\prime}+\eta \eta^{\prime}\right) \tag{4.15}
\end{equation*}
$$

similarly

$$
\begin{align*}
\frac{1}{\mathrm{p}_{62}^{\prime}} & =-\frac{1}{\mathrm{p}_{62}^{3}}\left(\xi \xi^{\prime}+b \xi^{\prime}+\eta \eta^{\prime}\right)  \tag{4.16}\\
\frac{1}{\mathrm{p}_{63}^{\prime}} & =-\frac{1}{\mathrm{p}_{63}^{3}}\left(\xi \xi^{\prime}-b \xi^{\prime}+\eta \eta^{\prime}\right)  \tag{4.17}\\
\frac{1}{\mathrm{p}_{64}^{\prime}} & =-\frac{1}{\mathrm{p}_{64}^{3}}\left(\xi \xi^{\prime}+a \xi^{\prime}+\eta \eta^{\prime}\right)  \tag{4.18}\\
\frac{1}{\mathrm{p}_{65}^{\prime}} & =-\frac{1}{\mathrm{p}_{65}^{3}}\left(\xi \xi^{\prime}-a \xi^{\prime}+\eta \eta^{\prime}\right) \tag{4.19}
\end{align*}
$$

Using equations from (4.10) to (4.19) in equation (4.9), we get

$$
\begin{align*}
& \frac{d}{d t}\left(\frac{1}{2} v^{2}-\frac{1}{2}\left(\xi^{2}+\eta^{2}\right)-m\left(\frac{1}{\mathrm{p}_{61}}\right)-m\left(\frac{1}{\mathrm{p}_{62}}\right)\right. \\
& \left.-m\left(\frac{1}{\mathrm{p}_{63}}\right)-M\left(\frac{1}{\mathrm{p}_{64}}\right)-M\left(\frac{1}{\mathrm{p}_{65}}\right)\right)=0 \tag{4.20}
\end{align*}
$$

Integrating equation (4.20) with respect to $\boldsymbol{t}$, we get:

$$
\begin{equation*}
\frac{1}{2} v^{2}-\frac{1}{2}\left(\xi^{2}+\eta^{2}\right)-m\left(\frac{1}{\mathrm{p}_{61}}+\frac{1}{\mathrm{p}_{62}}+\frac{1}{\mathrm{p}_{63}}\right)-M\left(\frac{1}{\mathrm{p}_{64}}+\frac{1}{\mathrm{p}_{65}}\right)=C . \tag{4.21}
\end{equation*}
$$

The constant C (named after the German mathematician Carl Jacobi who discovered it in 1836) is known as the Jacobi constant. In the collinear restricted six body problem, C is a constant of motion of $\boldsymbol{m}_{\mathbf{6}}$, here

- $-\frac{1}{\mathrm{p}_{61}},-\frac{1}{\mathrm{p}_{62}},-\frac{1}{\mathrm{p}_{63}},-\frac{1}{\mathrm{p}_{64}},-\frac{1}{\mathrm{p}_{65}}$ are the gravitational $\boldsymbol{P} . \boldsymbol{E}$ of the masses $\boldsymbol{m}_{\mathbf{1}}, \boldsymbol{m}_{\mathbf{2}}, \boldsymbol{m}_{\mathbf{3}}, \boldsymbol{m}_{\mathbf{4}}, \boldsymbol{m}_{\mathbf{5}}$ along horizontal axis.
- $\frac{1}{2} \boldsymbol{v}^{\mathbf{2}}$ is the $\boldsymbol{K} . \boldsymbol{E}$ per unit mass relative to rotating frame.
- $-\frac{1}{2}\left(\boldsymbol{\xi}^{2}+\eta^{2}\right)$ is the $\boldsymbol{P} . \boldsymbol{E}$ of the centrifugal force obtained by the rotation of the reference frame.

We rewrite the equation (4.21),

$$
\begin{equation*}
v^{2}=\left(\xi^{2}+\eta^{2}\right)+2 M\left(\frac{1}{\mathrm{p}_{64}}+\frac{1}{\mathrm{p}_{65}}\right)+2 m\left(\frac{1}{\mathrm{p}_{61}}+\frac{1}{\mathrm{p}_{62}}+\frac{1}{\mathrm{p}_{63}}\right)+2 C \tag{4.22}
\end{equation*}
$$

as $\boldsymbol{v}^{\mathbf{2}}$ cannot be negative, we can write true,

$$
\begin{equation*}
\left(\xi^{2}+\eta^{2}\right)+2 M\left(\frac{1}{\mathrm{p}_{64}}+\frac{1}{\mathrm{p}_{65}}\right)+2 m\left(\frac{1}{\mathrm{p}_{61}}+\frac{1}{\mathrm{p}_{62}}+\frac{1}{\mathrm{p}_{63}}\right)+2 C \geq 0 \tag{4.23}
\end{equation*}
$$

where

$$
\begin{equation*}
U(\xi, \eta)=\frac{\left(\xi^{2}+\eta^{2}\right)}{2}+M\left(\frac{1}{p_{64}}+\frac{1}{\mathrm{p}_{65}}\right)+m\left(\frac{1}{\mathrm{p}_{61}}+\frac{1}{\mathrm{p}_{62}}+\frac{1}{\mathrm{p}_{63}}\right) \tag{4.24}
\end{equation*}
$$

### 4.3 Equilibrium Solutions

The equations (4.4) and (4.5) do not have an analytical solution of a closed form, we can use these equations to determine the location of the equilibrium points. Equilibrium points are the places in space where the infinitesimal mass $\boldsymbol{m}_{\mathbf{6}}$ would have zero velocity and acceleration, i.e., where $\boldsymbol{m}_{6}$ appears at rest permanently relative to the masses $\boldsymbol{m}_{\mathbf{1}}, \boldsymbol{m}_{\mathbf{2}}, \boldsymbol{m}_{\mathbf{3}}, \boldsymbol{m}_{\mathbf{4}}, \boldsymbol{m}_{\mathbf{5}}$ respectively. These solutions can be found only if we meet the sufficient condition of all rates equal to zero,

$$
\xi^{\prime}=\eta^{\prime}=\xi^{\prime \prime}=\eta^{\prime \prime}=0
$$

To find the zeros $(\boldsymbol{\xi}, \boldsymbol{\eta})$ or equilibrium points / Lagrange point, we need to solve these equations numerically or drawing contour plot using Mathematica. The classification of equilibrium points for restricted collinear six body problem is discussed in the following cases:

### 4.4 Case 1: When $a \in[0.20,0.250], b=0.5$

There are two equilibrium points if we select $\boldsymbol{a}=\mathbf{0 . 2 0}$ as a marginal case, for $\boldsymbol{a} \in[\mathbf{0 . 2 0}, \mathbf{0 . 2 5 0}]$. If we choose any other value of $\boldsymbol{a}$ from the interval other
than $\mathbf{0 . 2 0}$ than there exits eight eqiulibrium points.
The blue contour line represents the contour of $\boldsymbol{U}_{\boldsymbol{\xi}}=\mathbf{0}$ and orange contour line represents the contour of $\boldsymbol{U}_{\eta}=\mathbf{0}$, respectively. The black dots show the position of mass and the red dots show the position of equilibrium points . (See Figures 4.6 to 4.9 below)

Equilibrium points, when $a=0.20, b=0.5$.
we take $\boldsymbol{a}=\mathbf{0 . 2 0}$ any point in the interval [ $\mathbf{0 . 2 0}, \mathbf{0 . 2 5 0}$ ], the corresponding values are $\boldsymbol{b}=0.5, \boldsymbol{m}=0.005$ and $\boldsymbol{M}=0.03219$.

Figure 4.1 shows that there are two equilibrium points $\boldsymbol{E}_{\mathbf{1}}, \boldsymbol{E}_{\mathbf{2}}$. They are collinear along $\boldsymbol{y}$-axis.
$\eta$


Figure 4.1: For $\mathbf{0 . 2 0} \leq \boldsymbol{a} \leq \mathbf{0 . 2 5 0}$, contour plot of $\boldsymbol{U}_{\boldsymbol{\xi}}=\mathbf{0}$ (blue) and $U_{\eta}=0$ (orange), when $a=0.20, b=0.5, m=0.00$ and $M=0.0363136$. Red dots represent the position of massess ( $\boldsymbol{m}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ ) and black dots represent position of equilibrium points $\left(\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}\right)$.

Equilibrium points, when $a=0.220, b=0.5$
We take $\boldsymbol{a}=\mathbf{0 . 2 2 0}$ to be any point in the interval, the corresponding value are $b=0.5, m=0.005$ and $M=0.03219$.

Figure 4.2 shows that there are eight equilibrium points. Two equilibrium points $\boldsymbol{E}_{\mathbf{1}}$ and $\boldsymbol{E}_{\mathbf{2}}$ are collinear along $\boldsymbol{y}$-axis and other six equilibrium points $\boldsymbol{E}_{\mathbf{4}}, \boldsymbol{E}_{\boldsymbol{5}}$, $\boldsymbol{E}_{6}, \boldsymbol{E}_{7}$, and $\boldsymbol{E}_{8}$ are collinear along $\boldsymbol{x}-\boldsymbol{a x i s}$.


Figure 4.2: For $\mathbf{0 . 2 0} \leq \boldsymbol{a} \leq \mathbf{0 . 2 5 0}$, contour plot of $\boldsymbol{U}_{\boldsymbol{\xi}}$ (blue) and $\boldsymbol{U}_{\boldsymbol{\eta}}$ (orange), when $a=0.220, b=0.5, m=0.005$ and $M=0.03219$. Red dots represent the position of equilibrium points $\left(\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\right)$, and black dots represent position of masses ( $\boldsymbol{m}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ )

Equilibrium points, when $a=0.233, b=0.5$

We take $\boldsymbol{a}=\mathbf{0 . 2 3 3}$ to be any point in the interval $[\mathbf{0 . 2 0 , 0 . 2 5 0 ]}$, the corresponding values are $\boldsymbol{b}=0.5, \boldsymbol{m}=0.018333$ and $\boldsymbol{M}=0.0256997$.

Figure 4.3 shows that there are eight equilibrium points. Two equilibrium points $\boldsymbol{E}_{\mathbf{1}}$ and $\boldsymbol{E}_{\mathbf{2}}$ are collinear along $\boldsymbol{y}-\boldsymbol{a x i s}$ and other six equilibrium points $\boldsymbol{E}_{\mathbf{4}}, \boldsymbol{E}_{\mathbf{5}}$, $\boldsymbol{E}_{\mathbf{6}}, \boldsymbol{E}_{\mathbf{7}}$, and $\boldsymbol{E}_{\mathbf{8}}$ are collinear along $\boldsymbol{x}$ - axis.


Figure 4.3: For $\mathbf{0 . 2 0} \leq \boldsymbol{a} \leq \mathbf{0 . 2 5 0}$, contour plot of $\boldsymbol{U}_{\boldsymbol{\xi}}$ (blue) and $\boldsymbol{U}_{\boldsymbol{\eta}}$ (orange), when $a=0.233, b=0.5, m=0.018333$ and $M=0.0256997$. Red dots represent the position of equilibrium points $\left(\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\right)$, and black dots represent position of masses ( $\boldsymbol{m}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ ).

Equilibrium points, when $a=0.244, b=0.5$
We take $\boldsymbol{a}=\mathbf{0 . 2 4 4}$ to be any point in the interval $[\mathbf{0 . 2 0}, \mathbf{0 . 2 5 0}]$, the corresponding values are $\boldsymbol{b}=0.5, \boldsymbol{m}=0.057222$ and $\boldsymbol{M}=0.0125336$.

Figure 4.4 shows that there are eight equilibrium points. Two equilibrium points
$\boldsymbol{E}_{\mathbf{1}}$ and $\boldsymbol{E}_{\mathbf{2}}$ are collinear along $\boldsymbol{y}$-axis and other six equilibrium points $\boldsymbol{E}_{\mathbf{4}}, \boldsymbol{E}_{\boldsymbol{5}}$, $\boldsymbol{E}_{\mathbf{6}}, \boldsymbol{E}_{\mathbf{7}}$, and $\boldsymbol{E}_{\mathbf{8}}$ are collinear along $\boldsymbol{x}$ - axis.


Figure 4.4: For $\mathbf{0 . 2 0} \leq \boldsymbol{a} \leq \mathbf{0 . 2 5 0}$, contour plot of $\boldsymbol{U}_{\boldsymbol{\xi}}$ (blue) and $\boldsymbol{U}_{\boldsymbol{\eta}}$ (orange), when $\boldsymbol{a}=0.244, \boldsymbol{b}=0.5, \boldsymbol{m}=\mathbf{0 . 0 5 7 2 2 2}$ and $\boldsymbol{M}=0.0125336$. Red dots represent the position of equilibrium points $\left(\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1 , 2 , 3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\right)$, and black dots represent position of masses ( $\boldsymbol{m}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ ).

### 4.5 Case 2: When $a \in[0.87,2.0], b=0.5$

There are four equilibrium points if we select $\boldsymbol{a}=\mathbf{0 . 8 7}$ as a marginal case, for $\boldsymbol{a} \in[\mathbf{0 . 8 7}, \mathbf{2 . 0}]$. If we choose any other value of $\boldsymbol{a}$ from the interval $[\mathbf{0 . 8 7}, \mathbf{2 . 0}]$ other than $\boldsymbol{a}=\mathbf{0 . 8 7}$ than there exits eight equilibrium points.

Equilibrium points, when $a=0.87, b=0.5$

We take $\boldsymbol{a}=\mathbf{0 . 8 7}$ to be any point in the interval $[\mathbf{0 . 8 7}, 2.0]$, the corresponding values are $\boldsymbol{b}=0.5, \boldsymbol{m}=0.1$ and $\boldsymbol{M}=0.00$.

Figure 4.5 shows that there are eight equilibrium points. Two equilibrium points $\boldsymbol{E}_{\mathbf{1}}$ and $\boldsymbol{E}_{\boldsymbol{2}}$ are collinear along $\boldsymbol{y} \boldsymbol{- a x i s}$, two equilibrium points $\boldsymbol{E}_{\mathbf{3}}$ and $\boldsymbol{E}_{\mathbf{4}}$ are collinear along $\boldsymbol{x}$-axis.


Figure 4.5: For $\mathbf{0 . 8 7} \leq \boldsymbol{a} \leq \mathbf{2 . 0}$, contour plot of $\boldsymbol{U}_{\boldsymbol{\xi}}$ (blue) and $\boldsymbol{U}_{\boldsymbol{\eta}}$ (orange), when $\boldsymbol{a}=\mathbf{0 . 8 7}, \boldsymbol{b}=\mathbf{0 . 5}, \boldsymbol{m}=\mathbf{0 . 1}$ and $\boldsymbol{M}=\mathbf{0 . 0 1 4 2 5 9 8}$. Red dots represent the position of equilibrium points $\left(\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\right)$, and black dots represent position of masses ( $m_{i}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ ).

Equilibrium points, when $a=1.08, b=0.5$
We take $\boldsymbol{a}=1.08$ to be any point in the interval, the corresponding values are $b=0.5, m=0.248762$ and $M=0.2989718$.

Figure 4.6 shows that there are eight equilibrium points. Two equilibrium points
$\boldsymbol{E}_{\mathbf{1}}$ and $\boldsymbol{E}_{\mathbf{2}}$ are collinear along $\boldsymbol{y}$-axis and other six equilibrium points $\boldsymbol{E}_{\mathbf{4}}, \boldsymbol{E}_{\mathbf{5}}$, $\boldsymbol{E}_{6}, \boldsymbol{E}_{7}$, and $\boldsymbol{E}_{8}$ are collinear along $\boldsymbol{x} \boldsymbol{- a x i s}$.


Figure 4.6: For $\mathbf{0 . 8 7} \leq \boldsymbol{a} \leq \mathbf{2 . 0}$, contour plot of $\boldsymbol{U}_{\boldsymbol{\xi}}$ (blue) and $\boldsymbol{U}_{\boldsymbol{\eta}}$ (orange), when $a=1.08, b=0.5, m=0.248762$ and $M=0.2989718$. Red dots represent the position of equilibrium $\operatorname{points}\left(\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\right)$, and black dots represent position of masses $\left(\boldsymbol{m}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\right)$.

Equilibrium points, when $a=1.32, b=0.5$
We take $\boldsymbol{a}=1.32$ to be any point in the interval, the corresponding values are $b=0.5, m=0.473939$ and $M=1.61644$.

Figure 4.7 shows that there are eight equilibrium points. Two equilibrium points $\boldsymbol{E}_{\mathbf{1}}$ and $\boldsymbol{E}_{\mathbf{2}}$ are collinear along $\boldsymbol{y}$-axis and other six equilibrium points $\boldsymbol{E}_{\mathbf{4}}, \boldsymbol{E}_{\mathbf{5}}$, $\boldsymbol{E}_{6}, \boldsymbol{E}_{7}$, and $\boldsymbol{E}_{8}$ are collinear along $\boldsymbol{x}-\boldsymbol{a x i s}$.


Figure 4.7: For $\mathbf{0 . 8 7} \leq \boldsymbol{a} \leq \mathbf{2 . 0}$, contour plot of $\boldsymbol{U}_{\boldsymbol{\xi}}$ (blue) and $\boldsymbol{U}_{\boldsymbol{\eta}}$ (orange), when $b=0.5 a=1.32, m=0.473939$ and $M=1.61644$. Red dots represent the position of equilibrium points $\left(\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\right)$, and black dots represent position of masses ( $\boldsymbol{m}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ ).

Equilibrium points, when $\boldsymbol{a}=1.98, \boldsymbol{b}=0.5$ We take $\boldsymbol{a}=1.98$ to be any point in the interval, the corresponding values are $\boldsymbol{b}=\mathbf{0 . 5}, \boldsymbol{m}=1.07242$ and $M=16.78364$.

Figure 4.8 shows that there are eight equilibrium points. Two equilibrium points $\boldsymbol{E}_{\mathbf{1}}$ and $\boldsymbol{E}_{\mathbf{2}}$ are collinear along $\boldsymbol{y}$-axis and other six equilibrium points $\boldsymbol{E}_{\mathbf{4}}, \boldsymbol{E}_{\boldsymbol{5}}$, $\boldsymbol{E}_{6}, \boldsymbol{E}_{7}$, and $\boldsymbol{E}_{8}$ are collinear along $\boldsymbol{x}-\boldsymbol{a x i s}$.


Figure 4.8: For $\mathbf{0 . 8 7} \leq \boldsymbol{a} \leq \mathbf{2 . 0}$, contour plot of $\boldsymbol{U}_{\boldsymbol{\xi}}$ (blue) and $\boldsymbol{U}_{\boldsymbol{\eta}}$ (orange), when $a=1.98, b=0.5, m=1.07242$ and $M=16.78364$. Red dots represent the position of equilibrium points $\left(\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}\right)$, and black dots represent position of masses ( $\boldsymbol{m}_{\boldsymbol{i}}, \boldsymbol{i}=\mathbf{1 , 2 , 3}, \mathbf{4}, \mathbf{5}$ ).

### 4.6 Stability Analysis of Equilibrium Points

This section is dedicated to the mathematical analysis of the stability of equilibrium points or lagrange points in R6BP. To check whether the equilibrium points or lagrangian points are stable or unstable, we perform an individual eigenvalue analysis for each equilibrium point.

Eigenvalues for Case-1:

Choose $\boldsymbol{a}=\mathbf{0 . 2 0}$ from $[\mathbf{0 . 2 0}, \mathbf{0 . 2 5 0}]$ and the corresponding values are $\boldsymbol{b}=\mathbf{0 . 5}$, $\boldsymbol{M}=0.0363136, \boldsymbol{m}=\mathbf{0 . 0 0}$ and $\boldsymbol{E}_{\mathbf{1}}(\mathbf{0 . 0}, \mathbf{0 . 3 6 2 )}$ (see figure 4.1), we will follow the procedure given in chapter 2 to analysis of stability. The Jacobian matrix is

$$
A=\left(\begin{array}{cc}
0.748956 & 0.00 \\
0.004 & 2.24685
\end{array}\right)
$$

The eigenvalues of matrix A are: $(\mathbf{2} .24685,0.748956)$, likewise, we have found eigenvalues for equilibrium points generated when $a=0.20,0.220,0.233,0.244,0.87,1.08,1.32,1.98$. We have shown coordinates of equilibrium points and their corresponding eigen values along with the stability status in Table 4.1 to $\mathbf{4 . 5}$ for different values of $\boldsymbol{a}$.

TABLE 4.1: Analysis of stability when $\boldsymbol{a}=\mathbf{0 . 2 0}, \boldsymbol{b}=\mathbf{0 . 5 0 0}$,

$$
M=0.0363136, m=0.00
$$

| Sr.No | Equilibrium points | Eigenvalues | Stability |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{E}_{\mathbf{1}}(0.0,0.362)$ | $(2.24685,0.748956)$ | unstable |
| 2 | $\boldsymbol{E}_{\boldsymbol{2}}(0.0,-0.362)$ | $(2.24685,0.748956)$ | unstable |

The same approach also applies to each equilibrium point, the eigenvalues are given below:

Table 4.2: Analysis of stability when $\boldsymbol{m}=\mathbf{0 . 0 0 5 4 6 2 9}, \boldsymbol{M}=\mathbf{0 . 0 3 2 1 9 0 3}$,

$$
a=0.220, b=0.500
$$

| Sr.No | Equilibrium points | Eigenvalues | Stability |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{E}_{\mathbf{1}}(0.0,0.362)$ | $(2.23531,0.767071)$ | unstable |
| 2 | $\boldsymbol{E}_{\mathbf{2}}(0.0,-0.362)$ | $(2.23531,0.7670716)$ | unstable |
| 3 | $\boldsymbol{E}_{\mathbf{3}}(-0.621,0.000459)$ | $(8.32723,-2.66361)$ | unstable |
| 4 | $\boldsymbol{E}_{\mathbf{4}}(0.621,0.000459)$ | $(8.32723,-2.66361)$ | unstable |
| 5 | $\boldsymbol{E}_{\mathbf{5}}(-0.408,0.004131)$ | $(25.0674,-11.0143)$ | unstable |
| 6 | $\boldsymbol{E}_{\mathbf{6}}(0.408,0.004131)$ | $(25.0674,-11.0143)$ | unstable |
| 7 | $\boldsymbol{E}_{\boldsymbol{7}}(-0.071,0.000459)$ | $(53.7948,-25.3966)$ | unstable |
| 8 | $\boldsymbol{E}_{\mathbf{8}}(0.071,0.000459)$ | $(53.7948,-25.3966)$ | unstable |

TABLE 4.3: Analysis of stability when $\boldsymbol{m}=\mathbf{0 . 0 1 8 3 3 3 6}, \boldsymbol{M}=\mathbf{0} .0256997$, $a=0.233, b=0.5$.

| Sr.No | Equilibrium points | Eigenvalues | Stability |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{E}_{\mathbf{1}}(0.002596,0.3884)$ | $(2.31164,0.69892)$ | unstable |
| 2 | $\boldsymbol{L}_{\mathbf{2}}(0.002596,-0.3884)$ | $(2.31164,0.69892)$ | unstable |
| 3 | $\boldsymbol{E}_{\mathbf{3}}(-0.695,0.001363)$ | $(6.66097,-1.83047)$ | unstable |
| 4 | $\boldsymbol{E}_{\mathbf{4}}(0.695,0.001363)$ | $(6.66097,-1.83047)$ | unstable |
| 5 | $\boldsymbol{E}_{\mathbf{5}}(-0.370,0.004089)$ | $(38.5901,-17.7694)$ | unstable |
| 6 | $\boldsymbol{E}_{\mathbf{6}}(0.370,0.004089)$ | $(38.5901,-17.7694)$ | unstable |
| 7 | $\boldsymbol{E}_{\mathbf{7}}(-0.105,0.001363)$ | $(59.2515,-28.1199)$ | unstable |
| 8 | $\boldsymbol{E}_{\mathbf{8}}(0.105,0.001363)$ | $(59.2515,-28.1199)$ | unstable |

TABLE 4.4: Analysis of stability when $\boldsymbol{m}=\mathbf{0 . 0 5 7 2 2 2}, \boldsymbol{M}=\mathbf{0 . 0 1 2 5 3 3 6}$,

$$
a=0.244, b=0.5, .
$$

| Sr.No | Equilibrium points | Eigenvalues | Stability |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{E}_{\mathbf{1}}(0.003083,0.4837)$ | $(2.38364,0.619541)$ | unstable |
| 2 | $\boldsymbol{E}_{\mathbf{2}}(0.003083,-0.4837)$ | $(2.38364,0.619541)$ | unstable |
| 3 | $\boldsymbol{E}_{\mathbf{3}}(-0.80,0.001785)$ | $(5.68192,-1.34095)$ | unstable |
| 4 | $\boldsymbol{E}_{\mathbf{4}}(0.80,0.001785)$ | $(5.68192,-1.34095)$ | unstable |
| 5 | $\boldsymbol{E}_{\mathbf{5}}(-0.330,0.001785)$ | $(67.1707,-32.0746)$ | unstable |
| 6 | $\boldsymbol{E}_{\mathbf{6}}(-0.330,0.001785)$ | $(67.1707,-32.0746)$ | unstable |
| 7 | $\boldsymbol{E}_{\mathbf{7}}(-0.1549,0.001785)$ | $(70.765,-33.8701)$ | unstable |
| 8 | $\boldsymbol{E}_{\mathbf{8}}(0.1549,0.001785)$ | $(70.765,-33.8701)$ | unstable |

TABLE 4.5: Analysis of stability when $\boldsymbol{m}=\mathbf{0 . 1 5 0 3 5 2 , ~} \boldsymbol{M}=\mathbf{0 . 0 1 4 2 5 9 8}$,
$a=0.249, b=0.5$.

| Sr.No | Equilibrium points | Eigenvalues | Stability |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{E}_{\mathbf{1}}(0.00357,0.65)$ | $(2.44069,0.567472)$ | unstable |
| 2 | $\boldsymbol{E}_{\mathbf{2}}(0.00357,-0.65)$ | $(2.44069,0.567472)$ | unstable |
| 3 | $\boldsymbol{E}_{\mathbf{3}}(-0.945,0.002336)$ | $(4.76689,-0.883443)$ | unstable |
| 4 | $\boldsymbol{E}_{\mathbf{4}}(0.945,0.002336)$ | $(4.76689,-0.883443)$ | unstable |
| 5 | $\boldsymbol{E}_{\mathbf{5}}(-0.325,0.001785)$ | $(4.16668,-2.00231)$ | unstable |
| 6 | $\boldsymbol{E}_{\mathbf{6}}(0.325,0.001785)$ | $(4.16668,-2.00231)$ | unstable |
| 7 | $\boldsymbol{E}_{\mathbf{7}}(-0.175,0.001785)$ | $(-5.11082,4.68278)$ | unstable |
| 8 | $\boldsymbol{E}_{\mathbf{8}}(0.175,0.001785)$ | $(-5.11082,4.68278)$ | unstable |

## Eigenvalues for Case-2 :

To check whether the equilibrium points or lagrangian points are stable or unstable, we perform an individual eigenvalue analysis for each equilibrium point. We have also conducted astability analysis for the Case-2 in which the interval for $\boldsymbol{a}$ is ( $0.87,2.0$ ). The coordinates of equilibrium points and corresponding eigen values along with the stability status are given in Table $4.6-4.9$ for different values of $\boldsymbol{a}$.

TABLE 4.6: Analysis of stability when $\boldsymbol{m}=\mathbf{0 . 1}, \boldsymbol{M}=\mathbf{0 . 0 1 4}$, $a=0.87, b=0.5$.

| Sr.No | Equilibrium points | Eigenvalues | Stability |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{E}_{\mathbf{1}}(0.00557,0.568)$ | $(2.40987,0.597351)$ | unstable |
| 2 | $\boldsymbol{E}_{\mathbf{2}}(0.00557,-0.568)$ | $(2.40987,0.597351)$ | unstable |
| 3 | $\boldsymbol{E}_{\mathbf{3}}(-0.8745,0.006328)$ | $(5.18135,-1.08024)$ | unstable |
| 4 | $\boldsymbol{E}_{\mathbf{4}}(0.8745,0.006328)$ | $(5.18135,-1.08024)$ | unstable |

There are four equilibrium points (see Table 4.6) and all theses equilibrium points are unstable. The points are $\boldsymbol{E}_{\mathbf{1}}(0.00557,0.568), \boldsymbol{E}_{\mathbf{2}}(0.00557,-0.568), \boldsymbol{E}_{\mathbf{3}}(-0.8745$, $0.006328), \boldsymbol{E}_{4}(0.8745,0.006328)$.

Table 4.7: Analysis of stability when $\boldsymbol{m}=\mathbf{0 . 2 4 8 7 6 2}, \boldsymbol{M}=\mathbf{0 . 2 9 8 9 7 1}$,

$$
a=1.08, b=0.5 .
$$

| Sr.No | Equilibrium points | Eigenvalues | Stability |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{E}_{\mathbf{1}}(0.004003,0.9031)$ | $(2.30876,0.692497)$ | unstable |
| 2 | $\boldsymbol{E}_{\mathbf{2}}(0.004003,-0.9031)$ | $(2.30876,0.692497)$ | unstable |
| 3 | $\boldsymbol{E}_{\mathbf{3}}(-1.59,0.00176)$ | $(6.34397,-1.67198)$ | unstable |
| 4 | $\boldsymbol{E}_{\mathbf{4}}(1.59,0.00176)$ | $(6.34397,-1.67198)$ | unstable |
| 5 | $\boldsymbol{E}_{\mathbf{5}}(-0.776,0.00528)$ | $(45.5469,-21.263)$ | unstable |
| 6 | $\boldsymbol{E}_{\mathbf{6}}(0.776,0.00528)$ | $(45.5469,-21.263)$ | unstable |
| 7 | $\boldsymbol{E}_{\mathbf{7}}(-0.2457,0.00528)$ | $(67.1595,-32.0584)$ | unstable |
| 8 | $\boldsymbol{E}_{\mathbf{8}}(0.2457,0.00528)$ | $(67.1595,-32.0584)$ | unstable |

Table 4.8: Analysis of stability when $\boldsymbol{m}=\mathbf{0 . 4 7 3 9 3 9 2}, \boldsymbol{M}=\mathbf{1 . 6 1 6 4 4}$, $a=1.32, b=0.5$.

| Sr.No | Equilibrium points | Eigenvalues | Stability |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{E}_{\mathbf{1}}(0.00,1.339)$ | $(2.17117,0.832416)$ | unstable |
| 2 | $\boldsymbol{E}_{\mathbf{2}}(0.00,-1.339)$ | $(2.17117,0.832416)$ | unstable |
| 3 | $\boldsymbol{E}_{\mathbf{3}}(-2.26,0.00)$ | $(5.37522,-1.18761)$ | unstable |
| 4 | $\boldsymbol{E}_{\mathbf{4}}(2.26,0.00)$ | $(5.37522,-1.18761)$ | unstable |
| 5 | $\boldsymbol{E}_{\mathbf{5}}(-0.776,0.00528)$ | $(68.0048,-32.4941)$ | unstable |
| 6 | $\boldsymbol{E}_{\mathbf{6}}(0.776,0.00528)$ | $(68.0048,-32.4941)$ | unstable |
| 7 | $\boldsymbol{E}_{\mathbf{7}}(-0.2457,0.00528)$ | $(128.039,-62.4788)$ | unstable |
| 8 | $\boldsymbol{E}_{\mathbf{8}}(0.2457,0.00528)$ | $(128.039,-62.4788)$ | unstable |

TABLE 4.9: Analysis of stability when $\boldsymbol{m}=\mathbf{1 . 0 7 2 4 2}, M=\mathbf{1 6 . 7 8 3 6}$, $a=1.988, b=0.5$.

| Sr.No | Equilibrium points | Eigenvalues | Stability |
| :---: | :---: | :---: | :---: |
| 1 | $\boldsymbol{E}_{\mathbf{1}}(0.0252,2.765)$ | $(2.11599,0.881011)$ | unstable |
| 2 | $\boldsymbol{E}_{\mathbf{2}}(0.0252,-2.765)$ | $(2.11599,0.881011)$ | unstable |
| 3 | $\boldsymbol{E}_{\mathbf{3}}(-4.159,0.01562)$ | $(4.52328,-0.761639)$ | unstable |
| 4 | $\boldsymbol{E}_{\mathbf{4}}(4.159,0.01562)$ | $(4.52328,-0.761639)$ | unstable |
| 5 | $\boldsymbol{E}_{\mathbf{5}}(-0.8409,0.01563)$ | $(83.0441,-39.9794)$ | unstable |
| 6 | $\boldsymbol{E}_{\mathbf{6}}(0.8409,0.01563)$ | $(83.0441,-39.9794)$ | unstable |
| 7 | $\boldsymbol{E}_{\boldsymbol{7}}(-0.25,0.01563)$ | $(286.796,-141.099)$ | unstable |
| 8 | $\boldsymbol{E}_{\mathbf{8}}(0.25,0.01563)$ | $(286.796,-141.099)$ | unstable |

### 4.7 Permitted Regions of Motion

One of the most important constants of dynamical system is the Jacobian constant ' C ' of motion, which represent the motion of the infinitesimal body. It can be used to sketch the regions of permitted motion. The boundaries between the prohibited and permitted regions are called zero velocity curves. In our geometry, we must now investigate these possibilities i.e., on the $\boldsymbol{x}$-axis, five masses are placed: $\boldsymbol{m}_{\mathbf{1}}, \boldsymbol{m}_{\mathbf{2}}, \boldsymbol{m}_{\mathbf{3}}, \boldsymbol{m}_{\mathbf{4}}$, and $\boldsymbol{m}_{\mathbf{5}}$, with the infinitesimal mass $\boldsymbol{m}_{\mathbf{6}}$
moving in the gravitational field of $\boldsymbol{m}_{\mathbf{1}}-\boldsymbol{m}_{\mathbf{5}}$. In Mathematica, we draw regions for different values of the Jacobian constant for Equation (4.24), and we obtain two regions, which are following:

1. Permissible region of motion (white area), where $\boldsymbol{m}_{\mathbf{6}}$ can freely move.
2. Shaded area (blue), where the motion of $\boldsymbol{m}_{6}$ is not allowed.

One can see easily that the Figures 4.9-4.34 by increasing the value of the Jacobian constant C from $\mathrm{C}=0.25$ to $\mathrm{C}=0.36$, the white region of motion of $\boldsymbol{m}_{6}$ is reducing and for $\mathrm{C}=26.0$ the masses $\boldsymbol{m}_{\mathbf{1}}, \boldsymbol{m}_{\mathbf{2}}, \boldsymbol{m}_{\mathbf{3}}, \boldsymbol{m}_{\mathbf{4}}, \boldsymbol{m}_{\mathbf{5}}$ are completely trapped, so for this, the value of C the $\boldsymbol{m}_{\mathbf{6}}$ can not reach around $\boldsymbol{m}_{\mathbf{1}}, \boldsymbol{m}_{\mathbf{2}}, \boldsymbol{m}_{\mathbf{3}}, \boldsymbol{m}_{\mathbf{4}}, \boldsymbol{m}_{\mathbf{5}}$.

### 4.7.1 Permitted Regions When $a=0.20$ and $b=0.5$

Figures 4.10 to $\mathbf{4 . 3 4}$ show the regions, where $\boldsymbol{m}_{\mathbf{6}}$ can move when $\boldsymbol{a}=\mathbf{0 . 2 0}$ and $\boldsymbol{b}=\mathbf{0 . 5}$. It is clearly visible that by increasing the value of ' C ' the permitted regions reduces.


Figure 4.9: Permitted regions of motion for $\mathrm{C}=0.25$.


Figure 4.10: Permitted regions of motion for $\mathrm{C}=0.30$.


Figure 4.11: Permitted regions of motion for $\mathrm{C}=0.31$.


Figure 4.12: Permissible regions of motion for $\mathrm{C}=0.36$.

### 4.7.2 Permitted Regions When $a=0.220$ and $b=0.5$



Figure 4.13: Permitted regions of motion for $\mathrm{C}=0.25$.


Figure 4.14: Permitted regions of motion for $\mathrm{C}=0.26$.


Figure 4.15: Permitted regions of motion for $\mathrm{C}=0.34$.


Figure 4.16: Permitted regions of motion for $\mathrm{C}=0.36$.

### 4.7.3 Permitted Regions When $a=0.233$ and $b=0.5$



Figure 4.17: Permitted regions of motion for $\mathrm{C}=0.4543$.


Figure 4.18: Permitted regions of motion for $\mathrm{C}=0.463$.


Figure 4.19: Permitted regions of motion for $\mathrm{C}=0.473$.

### 4.7.4 Permitted Regions When $a=0.244$ and $b=0.5$



Figure 4.20: Permitted regions of motion when $\mathrm{C}=0.77$.


Figure 4.21: Permitted regions of motion when $\mathrm{C}=0.75$.


Figure 4.22: Permitted regions of motion for $\mathrm{C}=0.80$.

### 4.7.5 Permitted Regions When $a=0.249$ and $b=0.5$



Figure 4.23: Permitted regions of motion for $\mathrm{C}=0.86$.


Figure 4.24: Permitted regions of motion for $\mathrm{C}=1.01$.


Figure 4.25: Permitted regions of motion for $\mathrm{C}=1.05$.

### 4.7.6 Permitted Regions When $a=1.08$ and $b=0.5$



Figure 4.26: Permitted regions of motion for $\mathrm{C}=2.50$.


Figure 4.27: Permitted regions of motion for $\mathrm{C}=2.85$.


Figure 4.28: Permitted regions of motion for $\mathrm{C}=3.1$.

### 4.7.7 Permitted Regions When $a=1.32$ and $b=0.5$



Figure 4.29: Permitted regions of motion when $\mathrm{C}=5.3$.


Figure 4.30: Permitted regions of motion for $\mathrm{C}=6.0$.


Figure 4.31: Permitted regions of motion for $\mathrm{C}=7.0$.

### 4.7.8 Permitted Regions When $a=1.98$ and $b=0.5$



Figure 4.32: Permitted regions of motion for $\mathrm{C}=19.9$.


Figure 4.33: Permitted regions of motion for $\mathrm{C}=24.9$.


Figure 4.34: Permitted regions of motion for $\mathrm{C}=26.0$.

## Chapter 5

## Conclusions

In this study we have investigated the motion of an infinitesimal mass under the gravitational influence of five large masses (primaries) is investigated. The primaries maintain collinear central configuration through out their motion. The pair of bigger masses are placed in the middles and the pair of smaller masses are placed at each corner and one smaller mass is at center of mass $(0,0)$. We have characterized the collinear central configuration and discovered that it holds for a fixed value of $\boldsymbol{b}=\mathbf{0 . 5}$, and for intervals $\mathbf{0 . 2 0} \leq \boldsymbol{a} \leq \mathbf{0 . 2 5 0}, 0.87 \leq \boldsymbol{a} \leq \mathbf{2 . 0}$. We sub-divided each intervals for $\boldsymbol{a}$ into four intervals where there is a clear change in position and number of equilibrium points. There are 2,4 , and 8 equilibrium points on the entire interval of $\boldsymbol{a}$ in various cases are observed. Using Mathematica stability analysis by applying the eigen value test for stability of equilibrium points is performed. According to this investigation, all the equilibrium points are unstable.By changing the value of Jacobian constant $\boldsymbol{C}$, the permissible region reduces and the prohibited region of motion of $\boldsymbol{m}_{\mathbf{6}}$ is also discussed.

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